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On Birkhoff's theorem in scalar–tensor theory of gravitation

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Abstract. It is shown that an analogue of Birkhoff's theorem of general relativity exists in a scalar–tensor theory of gravitation proposed by Sen and Dunn. Unlike in the Brans–Dicke scalar–tensor theory, Birkhoff's theorem is valid in the present theory irrespective of the nature of the scalar field introduced.

1. Introduction

Brans and Dicke (1961) have formulated a scalar–tensor theory of gravitation in which the tensor field alone is geometrized and the scalar field is alien to the geometry. Recently Sen and Dunn (1971) proposed a new scalar–tensor theory of gravitation in a modified riemannian manifold in which both the scalar and tensor fields have intrinsic geometrical significance. The scalar field, in this theory, is characterized by the function $x^0 = x^0(x^i)$ where x^i are coordinates in the four-dimensional Lyra manifold and the tensor field is identified with the metric tensor g_{ij} of the manifold.

The field equations given by Sen and Dunn for the combined scalar and tensor fields are

$$R_{ij}-\tfrac{1}{2}g_{ij}R = -8\pi G(x^0)^{-2}T_{ij}+\omega(x^0)^{-2}(x^0_{,i}x^0_{,j}-\tfrac{1}{2}g_{ij}x^0_{,k}x^{0,k}) \tag{1}$$

where $\omega = \frac{3}{2}$, T_{ij} is the energy–momentum tensor of the field and R is the usual Riemann curvature scalar. It was pointed out that these equations are identical with the Brans–Dicke equations, namely,

$$R_{ij}-\tfrac{1}{2}g_{ij}R = -8\Phi^{-1}T_{ij}+\omega\Phi^{-2}(\Phi_{,i}\Phi_{,j}-\tfrac{1}{2}g_{ij}\Phi_{,k}\Phi^{,k})+\Phi^{-1}(\Phi_{,i;j}-g_{ij}\Box\Phi) \tag{2}$$

$$\Box\Phi = \frac{8\pi T}{3+2\omega} \tag{3}$$

if the scalar function satisfied the condition

$$\Phi_{,i;j}-g_{ij}\Box\Phi = 0 \tag{4}$$

and $\omega = \frac{3}{2}$. However, the gravitational 'constant' must be redefined. While Sen and Dunn (1971) gave only a series type solution to the static vacuum field equations of the scalar–tensor theory in a Lyra manifold, Halford (1972) has obtained a closed-form exact solution and has shown that the present theory predicts the same effects within the observational limits as Einstein's theory.

In § 2 we discuss Birkhoff's theorem in the Brans–Dicke scalar–tensor theory of gravitation following Schücking (1957). In § 3, we show that the Birkhoff's theorem is valid, in the scalar–tensor theory proposed by Sen and Dunn, whatever may be the nature of the scalar field introduced.

2. Birkhoff's theorem in Brans–Dicke theory

It was shown by Birkhoff (1927) that every spherically symmetric solution of the Einstein vacuum field equations is static. This fact is known as Birkhoff's theorem. Schücking (1957) has shown that this theorem is valid in Jordan's (1952) extended theory of gravitation when the gravitational invariant of the theory is independent of time. On similar lines we show here, for completeness, that Birkhoff's theorem holds in the Brans–Dicke theory of gravitation when the scalar field introduced in the theory is independent of time.

We consider the spherically symmetric metric in the form

$$ds^2 = e^{\nu}\,dt^2 - e^{\lambda}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{5}$$

where

$$\lambda = \lambda(r, t); \qquad \nu = \nu(r, t) \tag{6}$$

with the scalar field

$$\Phi = \Phi(r, t). \tag{7}$$

The Brans–Dicke vacuum field equations for the metric (5) read as

$$\Phi\left[-e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}\right] = e^{-\nu}\left(\frac{\dot{\lambda}\dot{\Phi}}{2}-\frac{\omega\dot{\Phi}^2}{2\Phi}\right) - e^{-\lambda}\left(\Phi''-\frac{\lambda'\Phi'}{2}+\frac{\omega\Phi'^2}{2\Phi}\right) \tag{8}$$

$$\Phi\left[-e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right)+e^{-\nu}\left(\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\lambda}\dot{\nu}}{4}\right)\right] = e^{-\lambda}\left[\frac{\omega\Phi'^2}{2\Phi}-\frac{\Phi'}{r}\right] - e^{-\nu}\frac{\omega\dot{\Phi}^2}{2\Phi} \tag{9}$$

$$\Phi\left[e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}\right] = e^{-\lambda}\left(\frac{\omega\Phi'^2}{2\Phi}-\frac{\nu'\Phi'}{2}\right) + e^{-\nu}\left(\ddot{\Phi}+\frac{\omega\dot{\Phi}^2}{2\Phi}-\frac{\dot{\nu}\dot{\Phi}}{2}\right) \tag{10}$$

$$\Phi e^{-\lambda}\frac{\dot{\lambda}}{r} = e^{-\lambda}\left(\frac{\omega\dot{\Phi}'}{\Phi}+\dot{\Phi}'-\frac{\dot{\lambda}\Phi'}{2}-\frac{\nu'\dot{\Phi}}{2}\right) \tag{11}$$

$$\ddot{\Phi} - \tfrac{1}{2}(\dot{\nu}-\dot{\lambda})\dot{\Phi} = e^{\nu-\lambda}\left[\Phi''+\Phi'\left(\frac{2}{r}+\frac{\nu'-\lambda'}{2}\right)\right] \tag{12}$$

where primes denote partial differentiation with respect to r and dots denote partial differentiation with respect to t. When Φ is a constant this system reduces to the Einstein vacuum field equations in the spherically symmetric case and hence Birkhoff's theorem follows.

When the scalar field is a function of r alone, that is, $\dot{\Phi} = 0$, we have from (11) either

$$\dot{\lambda} = 0 \tag{13}$$

or

$$\Phi = \frac{\Phi_0}{r^2}, \qquad \Phi_0 = \text{constant} \tag{14}$$

when $\dot{\lambda} = 0$ it follows from (8), (10) and (12)

$$1 + \frac{r}{2}(\nu' - \lambda') + \frac{r\Phi'}{\Phi} = e^{\lambda}. \tag{15}$$

Partial differentiation of (15) with respect to t gives us

$$\dot{\nu}' = 0 \tag{16}$$

And when $\Phi = \Phi_0/r^2$ it follows again from (8) that

$$\nu' - \lambda' = \frac{2}{r}. \tag{17}$$

Substituting this in (15) we get

$$e^{\lambda} = 0. \tag{18}$$

Therefore, in this case, no solution arises. With this Birkhoff's theorem is proved in the Brans–Dicke theory of gravitation.

3. Birkhoff's theorem in scalar–tensor theory in a Lyra manifold

We consider the spherically symmetric line element in the form given by (5) with the scalar function

$$x^0 = x^0(r, t). \tag{19}$$

We shall prove that without assuming time independence of the scalar function x^0, the metric (5) is static in the present theory. The vacuum field equations of Sen and Dunn for the metric (5) can be written as

$$e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2} = \frac{\omega(x^0)^{-2}}{2}[e^{-\lambda}(x^{0\prime})^2+e^{-\nu}(\dot{x}^0)^2] \tag{20}$$

$$e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right)+e^{-\nu}\left(\frac{\ddot{\lambda}}{2}+\frac{\dot{\lambda}^2}{4}-\frac{\dot{\lambda}\dot{\nu}}{4}\right) = \frac{-\omega(x^0)^{-2}}{2}[e^{-\lambda}(x^{0\prime})^2-e^{-\nu}(\dot{x}^0)^2] \tag{21}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2} = \frac{\omega(x^0)^{-2}}{2}[e^{-\nu}(\dot{x}^0)^2+e^{-\lambda}(x^{0\prime})^2] \tag{22}$$

$$e^{-\lambda}\frac{\dot{\lambda}}{r} = 0. \tag{23}$$

From (23) we have

$$\dot{\lambda} = 0, \tag{24}$$

that is, λ is independent of time.

From (20) and (22) we have

$$1+\tfrac{1}{2}r(\nu'-\lambda') = e^{\lambda}. \tag{25}$$

Partial differentiation of (21) with respect to t gives us

$$\tfrac{1}{2}r(\dot{\nu}'-\dot{\lambda}') = e^{\lambda}\dot{\lambda}. \tag{26}$$

Using (24) in (26) we get

$$\dot{\nu}' = 0,$$

that is, ν is independent of time. Hence Birkhoff's theorem is valid in this theory whatever may be the nature of the scalar field.

4. Conclusions

In the scalar–tensor theory of Brans and Dicke no theorem analogous to Birkhoff's theorem in general relativity has been proved. Here we have shown that such a theorem is true, for the Brans–Dicke theory, in particular, when the scalar field is independent of time. But in the scalar–tensor theory proposed by Sen and Dunn Birkhoff's theorem is valid irrespective of the nature of the scalar field. Hence it may be considered that this theory is an improved version of the Brans–Dicke theory.

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